STUDIES IN THEORETICAL PHILOSOPHY

vol. 2

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Intuitionism vs. Classicism

A Mathematical Attack on Classical Logic

VITTORIO KLOSTERMANN

Bibliographische Information der Deutschen Nationalbibliothek
Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliographie; detaillierte bibliographische Daten sind im Internet über *http://dnb.dnb.de* abrufbar.

Gedruckt auf Alster Werkdruck der Firma Geese, Hamburg,
alterungsbeständig ∞ ISO 9706 und PEFC-zertifiziert.
Druck: betz-druck GmbH, Darmstadt
Printed in Germany
ISSN 2199-5214
ISBN 978-3-465-03906-8

For Esther

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Preface

This book is based on my dissertation *Intuitionistic Arguments Against Classical Logic*, which I wrote as a member of the Phlox research group between 2008 and 2012. For most of that time, I did not know I would write on this topic. Originally, I wanted to write about the semantics of explanation. My intent was to transfer techniques and results developed by relevant logicians to the semantical study of the sentential operator "because". However, this project failed. It took me quite a while to realise, but eventually I came to believe that implication and explanation are too dissimilar to use the achievements of relevant logicians for developing a semantics for "because". A longer period followed during which I had no clear idea which topic I wanted to take up instead, and, in 2009, I was happy to be distracted from philosophy by the birth of my first child and the ensuing chaos. When I started to do philosophy again, I realised that there was one aspect of the relevantists' project that puzzled me: They present a *coherent* criticism of *fundamental logical principles*, which I had always thought of as being too basic to allow for fruitful debate. I decided that on my second attempt I wanted to write about arguments against classical logic. Since I had (more or less) made up my mind about relevant logic already, I first turned to anti-classical arguments based on considerations about paradoxes of truth and then to the critique launched by mathematical intuitionists. The more I thought about intuitionism, the clearer it became to me that I wanted to focus my dissertation solely on this topic. Intuitionism is intimately related to fundamental questions about the nature of mathematics, epistemology, and semantics, and it stands out as one of a small number of enterprises whose adherents claim that they can rigorously establish the falsity of central classical logical principles. In my view, the conflict between classicists and intuitionists is exceptionally deep and far-reaching, and it is certainly one of the most radical disputes that can be pursued by rational means. Therefore, I am very happy to have been given the opportunity to develop my view on the nature of this argument.

Many friends and colleagues have made helpful comments on earlier drafts of my dissertation and this book. I am very grateful to all of them. In particular, I would like to thank: Sebastian Bünker, Miguel Hoeltje, Stephan Krämer, Raphael van Riel, Benjamin Schnieder, Moritz Schulz, Alexander Steinberg, and the members of Benjamin Schnieder's Research Colloquium in Hamburg, where I was able to discuss the second chapter of my dissertation.

A special thank you goes to the examiners of my dissertation: Benjamin Schnieder, Hannes Leitgeb, and Timothy Williamson. Benjamin has been a

great supervisor. In particular, he helped me with much understanding and support when I realised that I had to abandon my plan to write about the semantics of “because”. Hannes and Tim agreed, at very short notice, to examine my dissertation, for which I am infinitely grateful. The valuable comments provided by all three of them allowed me to write a much better book than I could have written without them.

I am indebted to the editors of this series: Tobias Rosefeldt and Benjamin Schnieder. It is a great privilege to have my book published in this series.

My greatest debt is to my family: my parents and siblings for all their help, my children Jannis and Kilian for the privilege of being able to spend so much time with them, and, most of all, my wife Esther, who has supported me much too much and who has prevented me from going crazy more than once during the last couple of years. While Leibniz was certainly wrong regarding the world as a whole, Esther, Jannis and Kilian are just as certainly a part of it that could not possibly be any better.

N.H.

Introduction

Logical reasoning is an especially secure means for obtaining knowledge. By logically inferring conclusions from known premises, we reliably come to know what was previously unknown to us, and, in various contexts, logical inferences are the *only* means by which we can extend our knowledge. *Mathematical truths* are particularly accessible items of human knowledge. While there is no guarantee that currently held empirical hypotheses remain to be found plausible in the future, one can be relatively confident that presently accepted mathematical statements will always be accepted by the mathematical community. The exceptional nature of logic and mathematics suggests that logical reasoning with purely mathematical concepts is one of the most reliable epistemic processes performed by humans. If someone does not accept a mathematical argument that is logically valid according to standard logical principles, the prospect for rational discussion may seem doubtful. According to Frege (1893: XVI), for example, such a person would simply be insane.

At the beginning of the previous century, Luitzen Egbertus Jan Brouwer declared standard mathematical practice and generally accepted forms of logical reasoning, referred to as *classical mathematics* and *classical logic*, to be seriously flawed. He propagated a new form of mathematics, called *intuitionism*, which incorporates principles that contradict basic assumptions of classical mathematics and even of classical logic. For example, Brouwer claimed that the following sentence is provable, although it is the negation of a classical logical truth that involves only logical and mathematical vocabulary:

Brouwer's Continuum Not every real number is or is not rational.

The nature and legitimacy of such mathematical and related philosophical claims are the topic of this book. My goal will be to reconstruct precise versions of intuitionistic arguments against classical logic, to analyse these arguments, and to evaluate them. I thereby hope to illustrate that it *is* possible to argue rationally even about the most fundamental principles of rational thought.

The first chapter focuses on logical theories and arguments against them. I will provide a general understanding of the notion of a logical theory and single out different types of arguments against a logical theory. In addition, I will take a look at two non-intuitionistic considerations which have been put forward against classical logic: considerations about relevance and considerations about empty singular terms. I will highlight those of their features which differentiate them from the anti-classical arguments that are brought up by intuitionists.

In the second chapter, I will discuss a famous claim about the nature of the dispute between classicists and intuitionists. According to Carnap, Quine, Dummett, and many others, classicists and intuitionists do not understand logical and mathematical vocabulary in the same way. It is claimed that logical and mathematical assertions of classicists and intuitionists are not contradictory. Their 'disagreement' is taken to be of a radically different nature than ordinary disagreements, in which one party affirms something that the other party denies. For example, when a classicist assertively utters the sentence "every real number is or is not rational", and an intuitionist assertively utters the sentence "not every real number is or is not rational", then, it is claimed, they do not contradict each other, because they do not attach the same meanings to the expressions "not", "or", "every", "real number", and "rational".

I will show that this claim is false. Drawing on results of Popper (1948) and Harris (1982), I will argue that classicists and intuitionists understand logical expressions in the same way. Furthermore, I will develop analogous results regarding mathematical terms and use them to show that classicists and intuitionists also understand these mathematical terms in the same way. The findings of Chapter 2 constitute an important precondition for the following chapters, in which intuitionistic arguments against classical logic are discussed. Since intuitionists attach the same senses to logical and mathematical expressions as classicists, their arguments, if cogent, would really show that classical logic and classical mathematics are incorrect.

In the first two sections of the third chapter, I will deal with two anti-classical arguments that are mathematical in character. Intuitionists develop their mathematics on the foundation of principles which contradict classical logical truths. I will introduce two such principles and discuss quasi-mathematical justifications for them that have recently been given by McCarty (2005) and de Swart (1992). The main objective will be to make the ideas of McCarty and de Swart sufficiently precise so that the most questionable premises of their reasoning can clearly be singled out.

In the third section of Chapter 3, I will introduce meta-mathematical arguments against classical logic. With those arguments, intuitionists try to show that certain basic classical principles are presently not known to be true by invoking assumptions about mathematical provability. The pertinent example will be an instance of the *principle of the excluded third*:

Goldbach's Disjunction Every or not every even number greater than 2 is the sum of two primes.

Intuitionists claim that Goldbach's Disjunction is presently not known to be true. Here, the crucial underlying thesis states that a mathematical disjunction is only provable if at least one of its disjuncts is provable, while, uncontroversially, neither of the disjuncts of Goldbach's Disjunction is known to be provable.

The question of how intuitionists might justify such a thesis leads to consideration of the notion of a mathematical proof and of the relation between mathematical activity and logical validity. This topic will be treated in the final section of the third chapter, in which I will discuss intuitionistic views about the relation between mathematics and logic. In particular, I will consider Brouwer's outlook on mathematical activity and Sundholm's (2004) views about legitimate inferential activity and the logical status of associated linguistic expressions. I will close Chapter 3 by presenting Prawitz' (2009, 2012a) anti-classical account of correct inferences and proofs. I will demonstrate that the fundamental principles of this account, as well as the other fundamental principles of the presented intuitionistic arguments, are in need of an independent justification if they are meant to undermine classical logic and mathematics. I will display that *semantic* considerations are the most promising candidates for such a justification.

The intuitionistic arguments in the third chapter are thus seen to lead to questions about the meanings of logical expressions and about the proper form of a semantic theory for a language that contains these expressions. In the fourth and fifth chapter, I will deal with intuitionistic arguments against classical logic which are explicitly meaning-theoretical in character. In recent philosophical discussions about the conflict between intuitionists and classicists, considerations about meanings and semantic theories play the most prominent role.

In the fourth chapter, I will discuss Dummett's famous *manifestation argument*. According to Dummett, the theory of classical logic depends on a truth conditional semantics that involves an epistemically unconstrained notion of truth. However, he takes the manifestation argument to show that a plausible truth conditional semantics must be based on an epistemically constrained notion of truth. The fundamental premises of the argument state that sentential understanding must be manifestable in linguistic behaviour, and that knowledge of truth conditions can only be manifested in such behaviour if the involved notion of truth is epistemically constrained.

The reasoning underlying Dummett's argument is rather complex, and it will be an important task to clarify its fundamental ideas and to rebut unconvincing refutations. In the final section of Chapter 4, I will show that the manifestation argument has to be rejected. It will be seen that it is possible to manifest one's knowledge of the truth condition of a sentence S by using the parts of S in various other sentences and that this behaviour does not require the ability to recognise the truth value of S in any possible situation. I will thus show that the manifestability of knowledge of truth conditions does not imply that the involved notion of truth is epistemically constrained.

In the fifth chapter, I will deal with *proof-theoretic arguments* against classical logic. These arguments combine semantic and epistemological considerations. Their proponents, e.g. Dummett (1991) and Tennant (1997), try to show that classical logic is false because it is not founded on a system of inferential rules that yields an account of the meanings of the logical operators. In addition to advancing the technical achievements of Read (2010), Humberstone (2011), and others, a major goal of Chapter 5 will be the development of a general understanding of proof-theoretic arguments and the precise reconstruction of the most important examples. It will be seen that disputes about the possible success of proof-theoretic arguments, such as the dispute between Dummett (1991) and Read (2000), have to be resolved by clarifying the general idea of a proof-theoretic argument. It will turn out that the plausibility of specific examples depends on subtle properties of systems of inferential rules that have not received due attention.

The most persuasive proof-theoretic arguments are based on the idea that the rules of a system that yields an account of the meanings of the logical operators have to exhibit a certain kind of *harmony*: the rules for introducing a logical operator must match the rules for eliminating it. Proponents of such proof-theoretic arguments claim that classical logic has to be rejected because no collections of harmonious rules of inference produce precisely those arguments that are logically valid according to classical logic. I will single out what I take to be the two most promising proof-theoretic arguments that are based on the premise that a sound logical theory must correspond to a collection of harmonious rules of inference. In the final section of Chapter 5, I will then show that both of these arguments must be rejected by presenting classical systems of harmonious rules of inference. Both proof-theoretic arguments will be seen to rely on an unjustified assumption about the precise form of an inferential account of the meanings the logical operators. Importantly, this assumption is independent of the requirement that a sound logical theory must correspond to a system of harmonious rules.

I will conclude the book with my view on the dispute between classicists and intuitionists: the only plausible arguments that either side can put forward against the other side are mathematical in character, but such arguments will always beg the question by presupposing the falsity of a fundamental logical or mathematical assumption of the view that is argued against; non-mathematical arguments, such as semantic, epistemological, or ontological ones, cannot be used to resolve the pertinent logical and mathematical disagreements. The dispute between classicists and intuitionists lacks a non-circular solution.

Chapter 1

Arguing about Classical Logic

The main subject of this book is the conflict between *classicists*, who endorse the predominant canon of logical methods, and *intuitionists*, who favour alternative logical principles that derive from Brouwer's mathematical innovations. Its principal aim consists in an evaluation of the intuitionistic challenges to classical logic. In preparation, I will discuss general features of arguments against classical logic. Against this background it will then be possible to highlight the peculiarities of the intuitionistic criticism.

In the first three sections, I will discuss the nature of arguments against classical logic in general. To this end, I will introduce the notion of a logical theory (1.1), present classical and intuitionistic first-order predicate logic (1.2), and discuss four ways of arguing against a logical theory (1.3). In the remaining two sections, I will present two non-intuitionistic considerations which have been put forward against classical logic: considerations about relevance (1.4) and considerations about empty singular terms (1.5). The discussion of these anti-classical arguments will be taken only to such a level that it is possible to compare them with those that are put forward by intuitionists.

1.1 Logical Theories

In one sense of the term, *logic* is one of the disciplines of pure mathematics, comprising such sub-disciplines as model theory, proof theory, and recursion theory. It is expressed in a mathematical language and pursued by mathematical methods. To argue against the standard execution of this discipline would involve a critique of the assumptions made and the methods used. Very often, however, arguments against classical logic are not directed against pure mathematics; they are based on a different conception of logic. In another sense of the term, *logic* is the theory of logical validity and related notions like logical truth and logical inconsistency. For what follows, this notion is the pertinent one. I will try to elucidate it in the present section.

1.1.1 The Bearers of Logical Properties

Given that the fundamental notions of logic are notions like logical truth and logical validity, it is desirable to know to which kinds of entities these notions apply. There are four main possibilities: (i) linguistic objects like (declarative) sentences and arguments, (ii) speech acts in which tokens of these linguistic objects are produced, (iii) mental acts like judgements and inferences, and (iv) abstract objects like propositions and propositional arguments. I will choose the first of these options, and I would like to indicate briefly my reasons for doing so. For simplicity, I will concentrate on sentences and their counterparts: sentential utterances, judgements, and propositions.

The strategic reason for assuming that some *sentences* are logically true is that there are sufficiently developed and widely accepted theories about them. These theories have been suggested by Tarski (1931), and they have been developed by Quine (1940: ch. 7) and Martin (1958: ch. 3). Speech acts and mental episodes, by contrast, have not received a comparable mathematical treatment, and propositions gave rise to numerous incompatible theories. There is also a non-strategic reason for preferring sentences over propositions (and, consequently, over judgements) in this context. Notions like logical truth seem to be sensitive to differences which, on the face of it, exist only at a linguistic level. Compare the following two sentences:

(1) All furze is furze. (2) All furze is gorse.

Although these two sentences are synonymous, only sentence (1) is a logical truth. If we make the reasonable assumption that they express the same proposition, then there is a logical difference which can only be accounted for at a linguistic level and not at a propositional level.[1]

For these reasons, I will assume that logic is primarily concerned with (declarative) sentences and other linguistic objects like arguments and theories. A fundamental assumption of the present approach is that for any natural language $\mathcal{L}$ there is the set of sentences of $\mathcal{L}$ which are logically true, the set of arguments of $\mathcal{L}$ which are logically valid, and the set of theories of $\mathcal{L}$ which are logically inconsistent. A logical theory is then put forward as a revealing description of these sets.

It has to be admitted that the choice of linguistic objects as the bearers of logical properties creates a number of problems. In particular, one has to mention the fact that expressions of natural languages may be structurally or

[1] The claim that (1) and (2) express the same proposition is based on three assumptions:

(a) The general terms "furze" and "gorse" are synonymous.

(b) If "furze" and "gorse" are synonymous, then so are (1) and (2).

(c) If (1) and (2) are synonymous, they express the same proposition (in a fixed context).

It should be noted that there are theories of propositions according to which (c) fails: theories according to which synonymous sentences may express distinct propositions in the same context (see Künne 2003: 369-72). Furthermore, there are theories of meaning according to which (b) fails: theories according to which substitution of synonymous parts (in extensional contexts) does not always preserve the meaning of the whole (see Fine 2007: 37-42).

lexically ambiguous and the fact that many of them exhibit various kinds of context-sensitivity. Consider the following candidate for a logically true sentence:

(3) If you are a philosopher, then you are a philosopher.

This conditional could be used to say something false when the two tokens of the pronoun "you" are used to refer to different persons (of whom only the first is a philosopher). It could also be uttered without asserting anything at all (for example, if it is used as a sample sentence of English without fixing a referent for the two tokens of "you"). In the end, an adequate logical theory should be applicable to context-sensitive expressions and thus will not simply be about sentences and other linguistic objects.[2] For simplicity, however, these complications will not be taken into account here.

Given the assumption that linguistic entities are the bearers of logical properties, it is helpful to single out one kind of linguistic entity and a corresponding logical notion. I take *arguments* to be the basic bearers of logical properties, and, consequently, I take the notion of *logical validity* to be the basic logical notion. I will assume that an *argument* consists of a finite list of sentences, its *premises*, a single sentence, its *conclusion*, and an expression like the word "therefore" which separates the list of premises from the conclusion.

As is well known, the notion of logical validity can be used to define several further logical notions such as logical implication, logical truth, and logical inconsistency. Importantly, however, there are also kinds of linguistic entities that fall under logical notions for which it is less clear how they are related to the notion of logical validity for arguments. In what follows, three closely related kinds of linguistic entities are of special relevance: *argumentations*, *argumentative steps*, and *rules* for performing argumentative steps. I will return to these objects and their logical properties in 1.2.5 and in later chapters.

1.1.2 Total and Partial Logical Theories

The following subsection introduces a distinction which relates to the notion of logical validity. What has been said thus far suggests that a logical theory for some language $\mathcal{L}$ is a theory which yields a specification of all logically valid arguments of $\mathcal{L}$. However, not every logical theory is like this. Consider the theory of classical sentential logic, and regard the following argument:

(4) Not every philosopher is wise. Therefore, some philosopher is not wise.

A proponent of classical sentential logic does not have to deny that this argument is logically valid. It would be a mistake though to infer that a logical theory only makes *positive* claims of logical validity. Classical sentential logic, for example, includes a *negative* thesis about (4). It treats a number of expressions, which typically include "not", "and", "or", and "if", as *logical constants*,

[2] Rumfitt (2007: 642), for example, proposes that such a theory is about what he calls *statements*, i.e. certain "ordered pairs whose first element is a meaningful, indeed disambiguated, declarative type-sentence and whose second element is a possible context of utterance".

and it involves the thesis that (4) is *not* validated by the logical properties of these constants.[3]

The moral to be drawn is that apart from what one might call a *total* logical theory for some language $\mathcal{L}$, which is put forward as an account of the totality of logically valid arguments of $\mathcal{L}$, there are also various *partial* logical theories for $\mathcal{L}$. Each of these theories may be put forward as an account of the class of arguments which are logically valid in $\mathcal{L}$ relative to a chosen set of logical constants. In fact, it suffices to consider partial logical theories since total logical theories can be treated as limiting cases of them: an argument is *logically valid* (*simpliciter*) iff it is logically valid with respect to the set of *all* logical constants.

1.1.3 The Shape of a Logical Theory

A logical theory specifies which arguments of the relevant language are logically valid in relation to the elements of the chosen set of logical constants. Let $\mathcal{L}$ be a natural language and suppose that $\mathcal{C}$ is a set of logical constants of $\mathcal{L}$. I will assume that a corresponding logical theory $\mathcal{T}$ is presented in a fragment of English. More precisely, the language of $\mathcal{T}$ is allowed to be an *extension* of a fragment of English; it may contain resources which are not part of English.

A logical theory $\mathcal{T}$ has to have a *syntactic part* $\mathcal{T}_{synt}$, a part for speaking about the syntactic properties of arguments of $\mathcal{L}$. This part should contain *canonical terms* for the expressions of $\mathcal{L}$, and it should contain syntactic predicates like "is an argument of $\mathcal{L}$". It will then consist of axioms which express basic principles about the syntactic properties of the expressions of $\mathcal{L}$. In addition to the syntactic part, the logical theory $\mathcal{T}$ needs to have a *main part*, a part for speaking about logical validity. I will assume that this part contains the unary predicate "is logically valid in $\mathcal{L}$ w.r.t. $\mathcal{C}$", which is meant to apply to those arguments of $\mathcal{L}$ that are logically valid in $\mathcal{L}$ with respect to the chosen set $\mathcal{C}$ of logical constants. The logical theory $\mathcal{T}$ is then the union of its syntactic part and its main part.

How could the syntactic part and the main part of a logical theory be stated? Given the syntactic complexity of natural languages this might seem extremely complicated. As a consequence, it might be thought that it is very difficult to compare competing logical theories. Fortunately, it is possible to deal at least with this second concern. For many pairs of competing logical theories it is possible to find surveyable simplifications of the pertinent natural languages such that the logical differences of the two theories can completely be stated with respect to these simplifications. The idea is to break up the main part of a logical theory into two parts: a *transformational part*, which reduces the syntactic complexity of the arguments through a formalisation process, and a *logical part*, which classifies the formalised versions of the original arguments (cp. Resnik 1985: 224 and Aberdein & Read 2009: 615-6).

[3] The notion of a logical constant might lack a sharp boundary. Furthermore, it should be emphasised that the assumption that there is a (fuzzy) line between logical and non-logical expressions does not imply that this distinction is epistemologically significant. See Field 2009a: 342-3 for skepticism about whether "the demarcation between logic and nonlogic" is philosophically important.

To make this explicit, consider a formal language $\mathcal{L}^*$ and assume that $\mathcal{T}_{synt^*}$ is a collection of axioms which express basic syntactic principles about the expressions of $\mathcal{L}^*$. In addition, suppose that $\mathcal{T}$ contains a unary function symbol π which stands for a *formalisation function* that maps every argument of $\mathcal{L}$ to a formal counterpart of $\mathcal{L}^*$. Furthermore, suppose that $\mathcal{T}$ contains the predicate "is $\mathcal{L}^*$-valid" for the formal arguments of $\mathcal{L}^*$ that correspond to the arguments of $\mathcal{L}$ which are logically valid in $\mathcal{L}$ with respect to $\mathcal{C}$. The language $\mathcal{L}$ is called the *informal object language*, and $\mathcal{L}^*$ is called the *formal object language*. The main part of $\mathcal{T}$ is now split up: first, there is a set of transformational axioms $\mathcal{T}_{trans}$ for π, which state how an argument is to be formalised; second, there is a set of axioms $\mathcal{T}_{logic}$ for the predicate "is $\mathcal{L}^*$-valid"; third, there is the following bridge principle, which yields the desired axiomatisation of the predicate "is logically valid in $\mathcal{L}$ w.r.t. $\mathcal{C}$":

Bridge Principle (BP) For every α, α is logically valid in $\mathcal{L}$ w.r.t. $\mathcal{C}$ if and only if $\pi(\alpha)$ is $\mathcal{L}^*$-valid.

This principle characterises the logically valid arguments of the informal object language as those that are mapped to $\mathcal{L}^*$-valid arguments of the formal object language. In sum, a logical theory $\mathcal{T}$ of this type is then the union of five parts:

$$\mathcal{T} = \mathcal{T}_{synt} \cup \mathcal{T}_{trans} \cup \mathcal{T}_{synt^*} \cup \{\text{BP}\} \cup \mathcal{T}_{logic}.$$

It has two syntactic parts, $\mathcal{T}_{synt}$ and $\mathcal{T}_{synt^*}$, which are linked via the transformational part $\mathcal{T}_{trans}$. In addition, it has a logical part, $\mathcal{T}_{logic}$, which is linked to the other parts by BP and which gives $\mathcal{T}$ its point.

I would like to emphasise that I do not consider the expressions of the formal object language $\mathcal{L}^*$ to be independently meaningful. They are only meaningful in a derivative sense according to which they inherit their meanings from their informal counterparts (see 2.1.2). Correspondingly, I consider logical validity to be fundamentally a property of arguments of the informal object language. The notion of $\mathcal{L}^*$-validity has only instrumental value, its job being to assist in axiomatising the fundamental notion of logical validity in $\mathcal{L}$ with respect to $\mathcal{C}$.

Dealing with logical theories of this type has one important advantage. In many cases competing logical theories can be taken to differ only in their logical parts. One can then restrict attention to the axioms for the predicate "is $\mathcal{L}^*$-valid", which are easier to survey than direct axioms for the predicate "is logically valid in $\mathcal{L}$ w.r.t. $\mathcal{C}$". In particular, the logical theories of classical and intuitionistic logic have the same syntactic parts and the same transformational part.[4] Thus, although the syntactic complexity of natural languages forces logical theories to be very complex, one does not have to take this complexity into account when one compares classical and intuitionistic logic.

It may be noted that the present conception of a logical theory is rather undemanding. In particular, its logical part is not required to involve axioms about entities like models or rules. This notion differs, therefore, from the notion

[4] The view that classical and intuitionistic logic have the same transformational part is partly based on the claim that classicists and intuitionists understand the relevant logical constants in the same way. This claim will be defended in Chapter 2.

of a logical theory put forward by Resnik (1985: 225) and the notion employed by Aberdein & Read (2009: 618). My reasons for adopting such an undemanding notion of a logical theory are purely pragmatic: it allows for rather simple theories of classical and intuitionistic logic. This will facilitate the discussion in the following chapters.

1.2 Classical and Intuitionistic Logic

I will now introduce the theories of classical and intuitionistic first-order predicate logic. These theories treat English as the informal object language; that is, they describe the logical properties of English. This means that the informal object language and the language of the logical theories coincide or, at least, overlap. Most of what follows can easily be adapted to cases in which these languages are disjoint. (For one exception, see the discussion of *basic criticism* in 1.3.2.) The elements of the following set and their stylistic variants are the logical constants of first-order predicate logic:

$$\mathcal{C} := \{\text{“not”}, \text{“or”}, \text{“and”}, \text{“if”}, \text{“every”}, \text{“some”}\}.$$

They will be called *standard logical constants*. Throughout this book, biconditionals, i.e. sentences of the form “S_1 if and only if S_2”, will be treated as conjunctions of conditionals.

First, I will introduce the formal object language $\mathcal{L}^*$ and indicate the formal syntactic part $\mathcal{T}_{synt^*}$, which is common to the logical theories discussed here (1.2.1). I will then present the logical part of the *model-theoretic* logical theory $\mathcal{T}^m$ (1.2.2), which is neutral between classical and intuitionistic logic, and the logical parts of two *derivational* logical theories (1.2.3): the classical logical theory $\mathcal{T}^C$ and the intuitionistic logical theory $\mathcal{T}^I$. Afterwards, I will discuss some aspects of the transformational part $\mathcal{T}_{trans}$ and the informal syntactic part $\mathcal{T}_{synt}$, which belong to the three introduced logical theories (1.2.4). Finally, I will remark on the relation between logically valid arguments and logically valid rules and their applications (1.2.5).

1.2.1 The Formal Syntactic Part

To begin with, the formal object language $\mathcal{L}^*$ will be specified. Its vocabulary consists of logical symbols, punctuation symbols, and non-logical symbols:

$$\neg \quad \vee \quad \wedge \quad \rightarrow \quad \forall \quad \exists \quad (\quad) \quad , \quad \succ \quad x \quad a \quad c \quad f \quad R \quad {}^\circ \quad *$$

The logical symbols $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\forall$, $\exists$ correspond to the standard logical constants. The punctuation symbols are brackets, the comma for building finite lists of sentences, and the symbol $\succ$ for building arguments from finite lists of sentences and sentences. The non-logical symbols x, a, c, f, R, $^\circ$, $*$ generate the parameters, variables,[5] constants, function signs, and relation signs. The

[5] I follow Prawitz (1965) in using parameters instead of allowing for free variables. The use of different kinds of symbols for (bound) variables and parameters will simplify the presentation of the logical calculi of natural deduction in 1.2.3.

symbol $^\circ$ is used to indicate the arities of function and relation signs, and $*$ is used to generate a sufficiently large supply of the different kinds of expressions. For example, the binary function signs are $f^{\circ\circ}$, $f^{\circ\circ}_{*}$, $f^{\circ\circ}_{**}$, *etc.* Whenever it is convenient, I will make informal use of further expressions as parameters, variables, constants, function signs, and relation signs; the context will make it clear to which category they belong.

The following standard syntactic notions are used in this text. Of fundamental importance is the notion of an expression of $\mathcal{L}^*$:

Expressions An *expression of* $\mathcal{L}^*$ is a finite concatenation of symbols of $\mathcal{L}^*$.

I will use the symbols of $\mathcal{L}^*$ as names for themselves, and I will use concatenations of symbols of $\mathcal{L}^*$ as terms for expressions of $\mathcal{L}^*$. The other syntactic notions defined here apply only to expressions of $\mathcal{L}^*$. The notions of an open term and of an open sentence are defined recursively:[6]

Open Terms A parameter is an open term. A constant is an open term. If $t_1, \ldots, t_n$ are open terms and f is an n-ary function sign, then $ft_1 \ldots t_n$ is an open term.

Open Sentences A sequence of an n-ary relation sign followed by n open terms is an open sentence. If S_1 and S_2 are open sentences, then so are $\neg S_1$, $(S_1 \wedge S_2)$, $(S_1 \vee S_2)$, $(S_1 \to S_2)$. If S_1 is an open sentence which contains tokens of the parameter x, and S_2 results from S_1 by substituting tokens of a variable a for the tokens of x, then $\exists a\, S_2$ and $\forall a\, S_2$ are open sentences, provided that the newly introduced tokens of a are bound by the initial quantifier.

Note that in an open sentence every variable is bound by some quantifier and every quantifier binds some variable. A *term* is an open term without parameters, and a *sentence* is an open sentence without parameters. Finally, the notions of an open list and of an open argument are defined as follows:

Open Lists An *open list* is a finite and alternating concatenation of open sentences and commas which begins and ends with an open sentence.

Open Arguments An *open argument* is either a concatenation of an open list, the symbol $\succ$, and an open sentence or the concatenation of the symbol $\succ$ and an open sentence.

For example, the expression $R^\circ x, \forall a\,(R^\circ a \to \neg R^\circ x) \succ \neg R^\circ c$ is an open argument. A *list* is an open list without parameters, and an *argument* is an open argument without parameters.

The formal syntactic part $\mathcal{T}_{synt^*}$ contains the fundamental syntactic assumptions about $\mathcal{L}^*$. They are assumptions about the identity and existence of expressions (see the end of 1.2.3). Importantly, the axioms of $\mathcal{T}_{synt^*}$ can be accepted by classicists and intuitionists. They belong to each of the three logical theories introduced here.

[6] In the following two definitions, f, x, and a are used as meta-variables.

1.2.2 A Model-Theoretic Logical Theory

In this and the following subsection, I will present the logical parts of three logical theories. First, I will introduce a *model-theoretic* approach originating with Tarski 1936. Then, I will present *derivational* approaches originating with Frege 1879 and Gentzen 1934. These or slightly different accounts can be found in any textbook on formal logic. I include them here for future reference.

The logical part of the model-theoretic logical theory $\mathcal{T}^m$ contains some purely set-theoretic axioms and a set MTA of model-theoretic axioms. I will indicate the axioms of MTA in an informal way here. To begin with, the notions of an M-term and of an M-sentence have to be introduced. Suppose that M is any set. Then, *M-terms* are certain finite sequences of symbols of $\mathcal{L}^*$ and elements of M: they are like open terms except that they contain elements of M instead of parameters.[7] One may think of M-terms of $\mathcal{L}^*$ as ordinary terms of an extension of $\mathcal{L}^*$: every element of M is added to $\mathcal{L}^*$ as a term for itself. *M-Sentences* are defined in analogy to M-terms: they are like open sentences except that they contain elements of M instead of parameters.

There is one axiom of MTA which contains the predicate "is $\mathcal{L}^*$-valid":

$\mathcal{L}^*$-Validity in Terms of Models

For every argument α of $\mathcal{L}^*$, α is $\mathcal{L}^*$-valid iff the conclusion of α is true in every model in which each premise of α is true.

In addition to some syntactic notions from $\mathcal{T}_{synt^*}$, this axiom presupposes the notion of a model and the notion of a sentence being true in a model. These notions are dealt with in additional model-theoretic axioms.

A *model* is a pair $\mathcal{M} := (M, I)$ which consists of an inhabited set M, its *domain*, and an *interpretation function* I.[8] A set is referred to as *inhabited* iff it has at least one element. According to classical logic, a set is inhabited iff it is not empty. However, intuitionists doubt this biconditional and prefer the displayed notion of a model (cp. Dummett 1991: 27). An *interpretation function* I is a function that maps every constant c to some element $c^{\mathcal{M}}$ of M, every n-place function sign f to some n-place function $f^{\mathcal{M}}$ on M, and every n-place relation sign R to some set $R^{\mathcal{M}}$ of n-tuples of M.

To define the notion of being true in a model, one first recursively defines the notion of a denotation function $D^{\mathcal{M}}$ from the set of M-terms of $\mathcal{L}^*$ to M:[9]

$$D^{\mathcal{M}}(m) := m, \quad D^{\mathcal{M}}(c) := c^{\mathcal{M}}, \quad D^{\mathcal{M}}(ft_1 \ldots t_n) := f^{\mathcal{M}}(D^{\mathcal{M}}(t_1), \ldots, D^{\mathcal{M}}(t_n)).$$

That is, every element of M denotes itself, every constant denotes its interpretation, and every term in which a function sign is attached to some terms denotes that element of M to which the interpretation of the function sign maps the denotations of the attached terms.

7 More precisely: an element of M is an M-term, a constant is an M-term, and if $t_1, \ldots, t_n$ are M-terms and f is an n-ary function sign, then $ft_1 \ldots t_n$ is an M-term.

8 To be precise, it has to be required that M does not contain variables or terms of $\mathcal{L}^*$.

9 In the following definition, $\mathcal{M} = (M, I)$ is a model, m is an element in M, c is a constant, f is an n-ary function sign, and $t_1, \ldots, t_n$ are M-terms. If τ is a function and x is an argument of τ, then $\tau(x)$ is the image of x under τ.

Then, a recursive definition of the predicate "is true in $\mathcal{M}$" is given:[10]

($\mathbf{Tr}_R^{\mathcal{M}}$) $Rt_1 \dots t_n$ is true in $\mathcal{M}$ iff $(D^{\mathcal{M}}(t_1), \dots, D^{\mathcal{M}}(t_n))$ is an element of $R^{\mathcal{M}}$;

($\mathbf{Tr}_\neg^{\mathcal{M}}$) $\neg S_1$ is true in $\mathcal{M}$ iff S_1 is not true in $\mathcal{M}$;

($\mathbf{Tr}_\rightarrow^{\mathcal{M}}$) $(S_1 \rightarrow S_2)$ is true in $\mathcal{M}$ iff S_2 is true in $\mathcal{M}$ if S_1 is true in $\mathcal{M}$;

($\mathbf{Tr}_\vee^{\mathcal{M}}$) $(S_1 \vee S_2)$ is true in $\mathcal{M}$ iff S_1 is true in $\mathcal{M}$ or S_2 is true in $\mathcal{M}$;

($\mathbf{Tr}_\wedge^{\mathcal{M}}$) $(S_1 \wedge S_2)$ is true in $\mathcal{M}$ iff S_1 is true in $\mathcal{M}$ and S_2 is true in $\mathcal{M}$;

($\mathbf{Tr}_\forall^{\mathcal{M}}$) $\forall aS$ is true in $\mathcal{M}$ iff, for every $m \in M$, S_a^m is true in $\mathcal{M}$;

($\mathbf{Tr}_\exists^{\mathcal{M}}$) $\exists aS$ is true in $\mathcal{M}$ iff, for some $m \in M$, S_a^m is true in $\mathcal{M}$.

Since every sentence is an M-sentence (for every M), this yields as a limiting case the desired relation between models and sentences.

Now, the model-theoretic axioms of MTA have to be supplemented with set-theoretic axioms. There are various strong collections of set-theoretic assumptions accepted by many intuitionists and classicists. The two most prominent such collections are IZF, the so-called *intuitionistic Zermelo-Fraenkel set theory*, and CZF, the so-called *constructivist Zermelo-Fraenkel set theory*. In classical logic, both IZF and CZF are equivalent to the standard classical Zermelo-Fraenkel set theory; in intuitionistic logic, CZF is weaker than IZF by conforming to *predicativist* standards.[11] In addition to the model-theoretic axioms, one may choose the axioms of either of these collections as the axioms of the logical part of $\mathcal{T}^m$. For definiteness, I will assume that the logical part of $\mathcal{T}^m$ equals CZF $\cup$ MTA. Importantly, the resulting logical theory $\mathcal{T}^m$ is accepted by most classicists and most intuitionists.

Neither IZF nor CZF comprises choice principles. There are weak choice principles, e.g. the axiom of *countable* choice or the axiom of *dependent* choice (see Troelstra & van Dalen 1988: 189-91), which many intuitionists accept. These could be added to the logical part of $\mathcal{T}^m$ as well.

There are also set-theoretic assumptions which are only accepted by one of the two camps. Famously, the *Axiom of Choice* is a principle not accepted by intuitionists,[12] while the so-called *Uniformity Principle* (see 3.1.1) is not accepted by classicists. Thus, there are theories which extend $\mathcal{T}^m$, some of which are rejected by intuitionists and some of which are rejected by classicists. (Of course, not every mathematician who accepts classical logic accepts the

[10] In the following definition, $\mathcal{M} = (M, I)$ is a model, m is an element of M, a is a variable, R is an n-place relation sign, $t_1, \dots, t_n$ are M-terms, S_1 and S_2 are M-sentences, and S is like an M-sentence, except that it contains *unbound* tokens of the variable a. In the last two clauses, S_a^m results from S by replacing every unbound token of a with m. If, for example, S is $(Fa \rightarrow \exists aGa)$, then S_a^m is $(Fm \rightarrow \exists aGa)$.

[11] Surveys of IZF are given in Beeson 1985: chs. 8 & 9 and in Ščedrov 1985. For CZF see Aczel 1978 and Aczel & Rathjen 2001.

[12] The Axiom of Choice has been introduced by Zermelo in his proof of the Well-Ordering Theorem (see Kanamori 2004). Diaconescu (1975) has discovered that the Axiom of Choice implies the principle of the excluded third; see also Goodman & Myhill 1978.

Axiom of Choice, and not every mathematician who accepts intuitionistic logic accepts the Uniformity Principle.)

It might be asked how it is possible that intuitionists and classicists disagree about which arguments are logically valid given that they accept the same logical theory $\mathcal{T}^m$. The simple answer is that they disagree about which sentences are consequences of $\mathcal{T}^m$. For example, a classicist will affirm while an intuitionist will deny that the following sentence is a theorem of $\mathcal{T}^m$:

(5) $\neg\neg Fc \succ Fc$ is $\mathcal{L}^*$-valid.

The reason for this is the classicist belief that for every model $\mathcal{M}$ and for every sentence S of $\mathcal{L}^*$, if S is not not true in $\mathcal{M}$, then S is true in $\mathcal{M}$, something which an intuitionist does not believe. To deal with this situation, it is helpful to consider different kinds of logical theories for which the different views about their consequences have less dramatic effects.

1.2.3 Two Derivational Logical Theories

The main alternative to a model-theoretic approach to logical validity is a derivational one. Here the situation is to some extent reversed: classicists and intuitionists sharply disagree about what kind of logical part a derivational theory should have, but they do not quarrel about which arguments are logically valid according to such a theory (see 1.3). Concepts from the derivational approach will play an important role in Chapters 2 and 5.

The logical part of the intuitionistic derivational theory $\mathcal{T}^I$ and the logical part of the classical derivational theory $\mathcal{T}^C$ each consists of a set of broadly syntactic, proof-theoretic axioms referred to as PTA^I and PTA^C. I will indicate them in an informal way here. (At the end of this subsection, I will mention how they can be stated more precisely.) To begin with, one needs to define a couple of new concepts. A *transition* is a pair consisting of a finite set of open arguments of $\mathcal{L}^*$, the *initial arguments*, and another open argument of $\mathcal{L}^*$, the *final argument*. A *rule* is a set of transitions, and a *calculus* is a collection of rules. The elements of a rule are also called its *applications*.[13] A calculus $\mathcal{R}$ induces the following property of arguments:

$\mathcal{R}$-Derivability An open argument α is called *$\mathcal{R}$-derivable* iff there is a finite sequence $\alpha_0, \ldots, \alpha_n$ of open arguments such that $\alpha = \alpha_n$ and such that for every $i \leq n$ there is a subset J_i of $\{0, 1, \ldots, i-1\}$ such that $(\{\alpha_j : j \in J_i\}, \alpha_i)$ is an application of a rule in $\mathcal{R}$.

According to this definition, an open argument is $\mathcal{R}$-derivable iff it can be obtained by a finite number of applications of rules in $\mathcal{R}$. A logical theory corresponding to a calculus $\mathcal{R}$ contains the following axiom:

13 It may be noted that rules are tied to a single (formal) language. In 2.2.3 and 2.2.4, I will argue for the philosophical importance of a different conception of rules, according to which they are language-transcendent objects.

$\mathcal{L}^*$-Validity in Terms of $\mathcal{R}$-Derivations (V$\mathcal{R}$)

For every argument α of $\mathcal{L}^*$, α is $\mathcal{L}^*$-valid iff α is $\mathcal{R}$-derivable.

That is, $\mathcal{L}^*$-validity is identified with $\mathcal{R}$-derivability.

There are two pertinent calculi: the intuitionistic calculus $\mathcal{R}^I$ and the classical calculus $\mathcal{R}^C$, giving rise to the axiom V$\mathcal{R}^I$ of PTA^I and to the axiom V$\mathcal{R}^C$ of PTA^C. The rules of the calculi $\mathcal{R}^I$ and $\mathcal{R}^C$ are introduced by presenting schemata for their applications. An application is written as follows:

$$\frac{t_1 \ldots t_n}{t}$$

where $t_1, \ldots, t_n$ represent the initial arguments, and t represents the final argument. In the following presentation, Γ, Δ, Σ are schematic letters for lists of open sentences, φ, ψ, χ are schematic letters for open sentences, and a, x, t are schematic letters for variables, parameters, and terms.

The intuitionistic calculus $\mathcal{R}^I$ comprises the following rules. First, there are three so-called *structural* rules, *Assumption*, *Exchange*, and *Contraction*, which are represented by schemata in which no logical operator occurs:

$$\frac{}{\varphi \succ \varphi}\ (\mathrm{A}) \qquad \frac{\Gamma, \varphi, \psi, \Delta \succ \chi}{\Gamma, \psi, \varphi, \Delta \succ \chi}\ (\mathrm{E}) \qquad \frac{\Gamma, \varphi, \varphi \succ \psi}{\Gamma, \varphi \succ \psi}\ (\mathrm{C})$$

Second, there are so-called *introduction* and *elimination rules*. For every logical operator there are one or two introduction rules and one or two elimination rules in whose schemata the operator figures:[14]

$$\frac{\Gamma \succ \varphi \quad \Delta \succ \psi}{\Gamma, \Delta \succ \varphi \wedge \psi}\ (\wedge_I) \qquad \frac{\Gamma \succ \varphi \wedge \psi}{\Gamma \succ \varphi}\ ({}^1\wedge_E) \qquad \frac{\Gamma \succ \varphi \wedge \psi}{\Gamma \succ \psi}\ ({}^2\wedge_E)$$

$$\frac{\Gamma \succ \varphi}{\Gamma \succ \varphi \vee \psi}\ ({}^1\vee_I) \qquad \frac{\Gamma \succ \psi}{\Gamma \succ \varphi \vee \psi}\ ({}^2\vee_I) \qquad \frac{\Gamma \succ \varphi \vee \psi \quad \Delta, \varphi \succ \chi \quad \Sigma, \psi \succ \chi}{\Gamma, \Delta, \Sigma \succ \chi}\ (\vee_E)$$

$$\frac{\Gamma, \varphi \succ \psi}{\Gamma \succ \varphi \rightarrow \psi}\ (\rightarrow_I) \qquad \frac{\Gamma \succ \varphi \rightarrow \psi \quad \Delta \succ \varphi}{\Gamma, \Delta \succ \psi}\ (\rightarrow_E)$$

$$\frac{\Gamma, \varphi \succ \psi \quad \Delta, \varphi \succ \neg\psi}{\Gamma, \Delta \succ \neg\varphi}\ (\neg_I) \qquad \frac{\Gamma \succ \varphi \quad \Delta \succ \neg\varphi}{\Gamma, \Delta \succ \psi}\ (\neg_E)$$

$$\frac{\Gamma \succ \varphi}{\Gamma \succ \forall a \varphi^a_x}\ (\forall_I) \qquad \frac{\Gamma \succ \forall a \varphi}{\Gamma \succ \varphi^t_a}\ (\forall_E)$$

$$\frac{\Gamma \succ \varphi^t_a}{\Gamma \succ \exists a \varphi}\ (\exists_I) \qquad \frac{\Gamma \succ \exists a \varphi \quad \Delta, \varphi^x_a \succ \psi}{\Gamma, \Delta \succ \psi}\ (\exists_E)$$

The rules $(\forall_I)$ and $(\exists_E)$ are subject to restrictions: in $(\forall_I)$, the parameter x must not occur in Γ, and in $(\exists_E)$, the parameter x must not occur in $\Gamma, \Delta, \exists a\varphi, \psi$.

[14] The schemata for the quantifier rules are to be understood as follows: in $(\forall_I)$, φ^a_x results from φ by replacing every token of x by a token of a, where it is required that every such new token of a is bound by the newly introduced initial quantifier; in $(\forall_E)$ and $(\exists_I)$, φ^t_a results from φ by replacing every token of a which is bound by the initial quantifier by a token of t; in $(\exists_E)$, φ^x_a results from φ by replacing every token of a which is bound by the initial quantifier by a token of x.